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On Perpetuants, with Applications to the Theory of Finite Quantics.

By J. Hammond.

1. The following formulæ are taken from two memoirs, both published in the *American Journal of Mathematics*, viz.:

Sylvester: On Subinvariants, *i. e.* Semi-Invariants to Binary Quantics of an Unlimited Order. (§4. Perpetuants, Vol. V, pp. 105–118.)

Cayley: A Memoir on Seminvariants (Vol. VII, pp. 1–25).

$$(2)=\frac{x^2}{\mathbf{2}},\quad (3)=\frac{x^3}{\mathbf{2}\cdot\mathbf{3}},\quad (4)=\frac{x^7}{\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}},\quad (5)=\frac{x^{15}}{\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}\cdot\mathbf{5}},$$

$$(2,\,2)=\frac{x^4}{\mathbf{2}\cdot\mathbf{4}},\quad (3,\,2)=\frac{x^5}{\mathbf{2}\cdot\mathbf{2}\cdot\mathbf{3}},\quad (4,\,2)=\frac{x^9}{\mathbf{2}\cdot\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}}$$

$$(3,\,3)=\frac{x^6+x^{11}}{\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}\cdot\mathbf{6}},\quad (2,\,2,\,2)=\frac{x^6}{\mathbf{2}\cdot\mathbf{4}\cdot\mathbf{6}}$$

$$(4)+(2,\,2)=\frac{x^4}{\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}}$$

$$(5)+(3,\,2)=\frac{x^5}{\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}\cdot\mathbf{5}}+S_5$$

$$(6)+(4,\,2)+(3,\,3)+(2,\,2,\,2)=\frac{x^6}{\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}\cdot\mathbf{5}\cdot\mathbf{6}}+S_6$$

The word "Perpetuant" with its corresponding idea, and most of the formulæ are due to Prof. Sylvester. The notation, in which the numbers $\mathbf{2},\,\mathbf{3},\,\ldots$, printed a little larger and thicker than usual, stand for $1-x^2$, $1-x^3$, . . . ; and the corrections S_5, S_6, corresponding to Quintic and Sextic Syzygies, belong to Prof. Cayley. Notice that, when m and n are unequal, we have $(m,\,n)=(m)\times(n)$; thus $(3,\,2)=(3)\times(2)$. But, if $(n)=\phi x$, $(n,\,n)=\dfrac{\varphi x^2+(\varphi x)^2}{2}$, thus

$$(3,\,3)=\tfrac{1}{2}\{\phi x^2+(\phi x)^2\}\text{ where }\phi x=\frac{x^3}{\mathbf{2}\cdot\mathbf{3}}$$

$$=\frac{1}{2}\left(\frac{x^6}{\mathbf{4}\cdot\mathbf{6}}+\frac{x^6}{\mathbf{2}\cdot\mathbf{2}\cdot\mathbf{3}\cdot\mathbf{3}}\right),$$ which, when reduced, gives the value written above.

The value of (3, 3) has, by a mere clerical error, been incorrectly given on p. 14 of Prof. Cayley's Memoir.

2. The syzygies denoted by S_5 and S_6 are irreducible, or ground-syzygies, and will be written $S_5^{(1)}$, $S_6^{(1)}$, for the sake of uniformity in what follows. But among the syzygies of degree seven are found the compounds (2) $S_5^{(1)}$, formed by multiplying the Quintic Syzygies by the Quadric Perpetuants, and these compounds are not all of them independent, but are connected with certain syzygies of the sixth degree by $S_7^{(2)}$ distinct relations, or syzygies of the second grade. And similar reasoning applies to the eighth degree. Hence

$$S_7 = S_7^{(1)} + (2)\, S_5^{(1)} - S_7^{(2)}$$
$$S_8 = S_8^{(1)} + (2)\, S_6^{(1)} + (3)\, S_5^{(1)} - S_8^{(2)}.$$

Moreover, in the formula for S_9 there will be compound syzygies of the second grade, in number (2) $S_7^{(2)}$, connected with certain of the $S_8^{(2)}$ by means of $S_9^{(3)}$ distinct relations, or syzygies of the third grade; and so on for higher degrees than the ninth.

We are not as yet provided with the means of obtaining generating functions for Perpetuants and Syzygies of degree n, or even of making the attempt with any chance of success; although Capt. MacMahon's recent researches (MacMahon: On Perpetuants, *American Journal of Mathematics*, Vol. VII, pp. 26–46 and pp. 256–263) have rendered it extremely probable that the generating function for Perpetuants of the n^{th} degree is

$$(n) = \frac{x^{2^{n-1}-1}}{2 \cdot 3 \cdot 4 \cdots n} \text{ when } n > 2.$$

This result will be confirmed for the cases $n = 6$ and $n = 7$, and the case $n = 5$ will be reconsidered as the simplest example of the method employed.

It seems that in general $S_n^{(\lambda)}$ (syzygies of degree n and grade λ) are not capable of being represented by any simple form of generating function, unless $n = 2\lambda + 3$, when by an extension of the method used, in the Johns Hopkins Circular for April, 1883 (p. 85), to find $S_5^{(1)}$, generating functions of a simple form may be obtained for the series of syzygies $S_5^{(1)}$, $S_7^{(2)}$, $S_9^{(3)}$, . . .

It should, however, be noticed that Prof. Cayley's Method of Capitation and Decapitation (see his memoir referred to) gives the simplest possible proof that the complete number of Quintic Syzygies has been found.

This will be sufficiently obvious if we remark that each congruence of the form $32^{\alpha-1}.\, 2^{\beta} \equiv 32^{\beta-1}.\, 2^{\alpha}$, when completed into a syzygy, contains a distinct irreducible Quartic Form; so that no repetition of the capitation process is possible.

Observe also that, if $2\alpha + 2\beta + 1 = w = 2n + 1$, the number of either the Quintic Syzygies, or of the special class of included Quartic Forms, is the same as the number of partitions of n into two unequal parts, zeros excluded. By means of these partitions I first arrived at the expression $\frac{x^7}{2\cdot 4}$ for the generating function of $S_5^{(1)}$, (see the J. H. U. Circular referred to).

3. Instead of $32^{\alpha-1}\ldots$ writing $C_{2\alpha+1}\ldots$, and $Q_{2\alpha}\ldots$ instead of $2^{\alpha}\ldots$, the general formula for Syzygies of the $S_5^{(1)}$ type is

$$C_{2\alpha+1}Q_{2\beta} - C_{2\beta+1}Q_{2\alpha} - a\Phi_{\alpha,\beta} = 0$$

where $\Phi_{\alpha,\beta}$ denotes the Quartic Form just mentioned, and the parent Quantic may be either $a + b\frac{x}{1} + c\frac{x^2}{1.2} + d\frac{x^3}{1.2.3} + \ldots$, or $(a, b, c, \ldots \between x, y,)^i$; but in the latter case i must not be less than either $2\alpha + 1$ or $2\beta + 1$. Hence the applications to the Theory of Finite Quantics.

It is easily seen that all Syzygies of the $S_7^{(2)}$ type (*i. e.* those of the seventh degree and second grade) are included in the formula

$$\begin{aligned}
& Q_{2\alpha}(C_{2\beta+1}Q_{2\gamma} - C_{2\gamma+1}Q_{2\beta} - a\Phi_{\beta,\gamma}) \\
-\ & Q_{2\beta}(C_{2\alpha+1}Q_{2\gamma} - C_{2\gamma+1}Q_{2\alpha} - a\Phi_{\alpha,\gamma}) \\
+\ & Q_{2\gamma}(C_{2\alpha+1}Q_{2\beta} - C_{2\beta+1}Q_{2\alpha} - a\Phi_{\alpha,\beta}) \\
+\ & a\,(Q_{2\alpha}\Phi_{\beta,\gamma} - Q_{2\beta}\Phi_{\alpha,\gamma} + Q_{2\gamma}\Phi_{\alpha,\beta}) = 0
\end{aligned}$$

connecting compounds of the type $(2)\,S_5^{(1)}$ with a special class of Sextic Syzygies, and that, if $2\alpha + 2\beta + 2\gamma + 1 = w = 2n + 1$, their number is the same as that of the partitions of n into three unequal parts, zeros excluded.

This may by a convenient abbreviation be written

$$Q_{2\alpha}\overline{\beta\gamma} - Q_{2\beta}\overline{\alpha\gamma} + Q_{2\gamma}\overline{\alpha\beta} + a\,.\,\overline{\alpha\beta\gamma} = 0$$

and there is a similar formula for some of the syzygies of the type $S_8^{(2)}$, viz.

$$C_{2\alpha+1}\overline{\beta\gamma} - C_{2\beta+1}\overline{\alpha\gamma} + C_{2\gamma+1}\overline{\alpha\beta} + a\,(C_{2\alpha+1}\Phi_{\beta,\gamma} - C_{2\beta+1}\Phi_{\alpha,\gamma} + C_{2\gamma+1}\Phi_{\alpha,\beta}) = 0\,;$$

but this, unlike the former, will not give the complete set of $S_8^{(2)}$ syzygies, because for the eighth degree the compounds are of two types $(2)\,S_6^{(1)}$ and $(3)\,S_5^{(1)}$ (*vide* Art. 2). In like manner all the $S_9^{(3)}$ syzygies (and a second special class of $S_8^{(2)}$ syzygies of odd weight) are given by the formula,

$$\begin{aligned}
& Q_{2\alpha}(Q_{2\beta}\overline{\gamma\delta} - Q_{2\gamma}\overline{\beta\delta} + Q_{2\delta}\overline{\beta\gamma} + a\overline{\beta\gamma\delta}) \\
-\ & Q_{2\beta}(Q_{2\alpha}\overline{\gamma\delta} - Q_{2\gamma}\overline{\alpha\delta} + Q_{2\delta}\overline{\alpha\gamma} + a\overline{\alpha\gamma\delta}) \\
+\ & Q_{2\gamma}(Q_{2\alpha}\overline{\beta\delta} - Q_{2\beta}\overline{\alpha\delta} + Q_{2\delta}\overline{\alpha\beta} + a\overline{\alpha\beta\delta}) \\
-\ & Q_{2\delta}(Q_{2\alpha}\overline{\beta\gamma} - Q_{2\beta}\overline{\alpha\gamma} + Q_{2\gamma}\overline{\alpha\beta} + a\overline{\alpha\beta\gamma}) \\
-\ & a\,(Q_{2\alpha}\overline{\beta\gamma\delta} - Q_{2\beta}\overline{\alpha\gamma\delta} + Q_{2\gamma}\overline{\alpha\beta\delta} - Q_{2\delta}\overline{\alpha\beta\gamma}) = 0
\end{aligned}$$

connecting compounds of the type $(2)\,S_7^{(2)}$ with the special class of $S_8^{(2)}$ syzygies, and if $2\alpha + 2\beta + 2\gamma + 2\delta + 1 = w = 2n + 1$, their number is that of the partitions of n into four unequal parts, zeros excluded.

This formula also may be abbreviated, by writing a single symbol for each syzygy of the second grade contained in it, *i. e.* for each expression in ( ); and when this is done, it will be easy to write down the corresponding formula for all the $S_{11}^{(4)}$ syzygies, which will be found to connect five compounds of the type $(2)\,S_9^{(3)}$ with one of the type $aS_{10}^{(3)}$ by a linear relation. And just as the $S_8^{(2)}$ syzygies were shown to form a special class only, and not the complete set, because of the existence of compounds of the two types $(2)\,S_6^{(1)}$ and $(3)S_5^{(1)}$; so the $S_{10}^{(3)}$ will form a special class because of the existence of the two types of compounds $(2)\,S_8^{(2)}$ and $(3)\,S_7^{(2)}$. The $S_{11}^{(4)}$ on the other hand will be completely determined (just as $S_9^{(3)}$ and $S_7^{(2)}$ were), their number being that of the partitions of $n = \frac{1}{2}(w - 1)$ into five unequal parts, zeros excluded.

Precisely similar reasoning holds in every case, and leads to the following THEOREM:—*All the $S_{2\lambda+3}^{(\lambda)}$ syzygies are of odd weight $(w = 2n + 1)$, and connect compounds of the type $(2)\,S_{2\lambda+1}^{(\lambda-1)}$ with a special class of $S_{2\lambda+2}^{(\lambda-1)}$ syzygies, their number being that of the partitions of n into $\lambda + 1$ unequal parts, zeros excluded. But for a finite Quintic of order i, none of these parts must exceed $\frac{i-1}{2}$ if i be odd, or $\frac{i}{2} - 1$ if i be even.*

4. Let $\binom{n}{k}$ denote the number of partitions of n into k unequal parts, zeros excluded. By the subtraction of unity from each part we obtain the partitions of $n - k$ into k unequal parts, zeros included; and these are the same as the partitions of $n - k$ into k unequal parts, zeros excluded, + those of $n - k$ into $k - 1$ unequal parts, zeros also excluded. This is the meaning of the formula

$$\binom{n}{k} = \binom{n-k}{k} + \binom{n-k}{k-1}.$$

Now the theorem is that

$$S_{2\lambda+3}^{(\lambda)} = \Sigma \binom{n}{\lambda+1} x^{2n+1}.$$

Whence $$(1 - x^{2\lambda+2})\,S_{2\lambda+3}^{(\lambda)} = \Sigma \left\{ \binom{n}{\lambda-1} - \binom{n-\lambda-1}{\lambda+1} \right\} x^{2n+1}.$$

The above partition formula reduces this to $\Sigma \binom{n-\lambda-1}{\lambda} x^{2n+1}$, or to $\Sigma \binom{m}{\lambda} x^{2m+1+2\lambda+2}$ if we write $n = m + \lambda + 1$. So that finally we obtain

$$S^{(\lambda)}_{2\lambda+3} = \frac{x^{2\lambda+2}}{1-x^{2\lambda+2}} S^{(\lambda-1)}_{2\lambda+1} = \frac{x^{\lambda^2+3\lambda+3}}{2\cdot4\cdot6\cdot\ldots(\lambda+1)\text{ factors}}.$$

Particular cases are

$$\left.\begin{array}{l} S^{(1)}_{5} = \dfrac{x^{7}}{2\cdot4} \\ S^{(2)}_{7} = \dfrac{x^{13}}{2\cdot4\cdot6} \\ S^{(3)}_{9} = \dfrac{x^{21}}{2\cdot4\cdot6\cdot8} \\ S^{(4)}_{11} = \dfrac{x^{31}}{2\cdot4\cdot6\cdot8\cdot10} \\ \ldots\ldots\ldots\ldots \end{array}\right\}$$

Postponing, for the present, the application to Finite Quantics, I remark that Syzygies of each of these four types are recognizable in the Numerator of the Rep. Generating Function for the Binary 12ic, the type $S^{(4)}_{11}$ being represented by a single term. (*Vide* pp. 44 and 45 of Sylvester's Tables of the Generating Functions and Groundforms of the Binary Duodecimic, with remarks, &c., *Am. Journ.*, Vol. IV, pp. 41 *et seq.*)

and it should always be borne in mind that each of these generating functions will also serve to express the number of syzygies of a special class whose degree and grade are each lower by unity than those indicated. Thus the same generating function serves to express the total number of $S^{(2)}_{7}$ syzygies, and that of a special class of $S^{(1)}_{6}$ syzygies.

5. A simple generating function may be defined as one of the form $\dfrac{x^{w}}{(1-x^{\lambda})(1-x^{\mu})(1-x^{\nu})\ldots}$, where $\lambda, \mu, \nu, \ldots$ are all different.

It corresponds to all the partitions obtainable from a single partition of w, denoted by x^{w}, by superadding any number of each of the parts $\lambda, \mu, \nu, \ldots$ This partition of w may be perfectly determinate or only partially so; but in the case of perpetuants, here considered, both its parts and the superadded ones $\lambda, \mu, \nu, \ldots$ must be non-unitary. A good example is the generating function for Quintic Perpetuants where $w = 15$, the superadded parts being 2, 3, 4, and 5, and the partition of w may be any one of the three 555, 5532, or 5433.

A first reduction of the formulæ for perpetuants is obtained by using only those simple generating functions in which $\lambda, \mu, \nu, \ldots$ are all even and the partition of w contains a known number of odd parts; the other generating functions used being expanded in a series of these.

Thus in the table, which gives the first and only reduction for Quintic Perpetuants; all the simple generating functions in the n^{th} row correspond to partitions containing n odd parts. Columns I and II are obtained by decomposition, as explained below, from $\frac{x^5}{2\cdot3\cdot4\cdot5}$ and $\frac{x^5}{2\cdot2\cdot3}$; and III by the row from row subtraction of II from I. The addition of the positive terms of III gives $\frac{x^{15}}{2\cdot3\cdot4\cdot5}$, or the generating function of Quintic Perpetuants; the negative term is the generating function of Quintic Syzygies.

	I	II	III
1	$\frac{x^5}{2\cdot4}$	$\frac{x^5+x^7}{2\cdot4}$	$-\frac{x^7}{2\cdot4}$
2	$\frac{x^8+x^{10}}{2\cdot4}$	$\frac{x^8+x^{10}}{2\cdot4}$	0
3	$\frac{x^{11}+x^{13}+x^{15}}{2\cdot4}$	$\frac{x^{11}+x^{13}}{2\cdot4}$	$\frac{x^{15}}{2\cdot4}$
4	$\frac{x^{14}+x^{16}+x^{18}+x^{20}}{2\cdot4}$	$\frac{x^{14}+x^{16}}{2\cdot4}$	$\frac{x^{18}+x^{20}}{2\cdot4}$
etc.	etc.	etc.	etc.

A system of partitions of the type $5^{A+1}4^B3^C2^D$ can be decomposed into partitions of the types

$$\begin{array}{lllll} 54^B2^D, & 54^B32^D, & 54^B3^22^D, & 54^B3^32^D, & \text{etc.} \\ & 5^24^B2^D, & 5^24^B32^D, & 5^24^B3^22^D, & \text{etc.} \\ & & 5^34^B2^D, & 5^34^B32^D, & \text{etc.} \\ & & & 5^44^B2^D, & \text{etc.} \end{array}$$

and to each of these corresponds a term of the decomposition of the generating function of $5^{A+1}4^B3^C2^D$, *i. e.* a term of

$$\frac{x^5}{2\cdot3\cdot4\cdot5}=\frac{x^5}{2\cdot4}+\frac{x^8+x^{10}}{2\cdot4}+\frac{x^{11}+x^{13}+x^{15}}{2\cdot4}+\frac{x^{14}+x^{16}+x^{18}+x^{20}}{2\cdot4}+\text{etc.}$$

where the simple generating functions of partitions with the same number of odd parts have been grouped together, as in Col. I of the table.

To obtain Col. II notice that products of the type (3, 2) or $3^{\alpha+1}2^{\beta}.2^{\gamma+1}$ can be decomposed into the products $32^{\beta}.2^{\gamma+1}$, $3^{2}2^{\beta}.2^{\gamma+1}$, $3^{3}2^{\beta}.2^{\gamma+1}$, $3^{4}2^{\beta}.2^{\lambda+1}$, etc. Or passing to their corresponding generating functions

$$\frac{x^5}{2\cdot 2\cdot 3}=\frac{x^5}{2\cdot 2}+\frac{x^8}{2\cdot 2}+\frac{x^{11}}{2\cdot 2}+\frac{x^{14}}{2\cdot 2}+\dots$$

The transition from this to Col. II of the table, preparatory to its ultimate subtraction from Col. I, though simple and even trifling in appearance, deserves some notice. It signifies that each of the products spoken of has to be transformed, by means of Prof. Cayley's Multiplication Theory, into a series of *simple partitions;* and when this has been done, we have a system of equations for the reduction of the partitions of Col. I. The unreduced partitions now correspond to the positive terms of Col. III, which, when collected, constitute the Generating Function for Quintic Perpetuants; the reduced partitions are made to depend on these and on the products (3, 2). To see the exact meaning of the negative term in row 1, or the Generating Function for Quintic Syzygies, observe that the products $32^{\beta}.2^{\gamma+1}$ may be divided into three sets, viz.

(*a*) those for which $\beta=\gamma$, generating function $\dfrac{x^5}{4}$
(*b*) " " $\beta<\gamma$, " " $\dfrac{x^7}{2\cdot 4}$
(*c*) " " $\beta>\gamma$, " " $\dfrac{x^7}{2\cdot 4}$

and that either of the sets (*b*) or (*c*) taken with (*a*) being capable of reducing all the partitions of one odd part in Col. I, the other set is incapable of giving any fresh reductions and therefore yields syzygies.

6. Reasoning precisely similar to that employed in the preceding article gives the decomposition

$$\frac{x^6}{2\cdot 3\cdot 4\cdot 5\cdot 6}=\frac{x^6}{2\cdot 4\cdot 6}+\frac{x^9+x^{11}}{2\cdot 4\cdot 6}+\frac{x^{12}+x^{14}+x^{16}}{2\cdot 4\cdot 6}$$
$$+\frac{x^{15}+x^{17}+x^{19}+x^{21}}{2\cdot 4\cdot 6}+\frac{x^{18}+x^{20}+x^{22}+x^{24}+x^{26}}{2\cdot 4\cdot 6}$$
$$+\frac{x^{21}+x^{23}+x^{25}+x^{27}+x^{29}+x^{31}}{2\cdot 4\cdot 6}+\text{etc.}$$

where we may at once dismiss the first term with the remark that it corresponds to partitions with no odd part, and is therefore exactly reducible by means of the products (2, 2, 2). The remaining terms, with their common denominator omitted for the sake of brevity, are arranged in the first column of the table.

Den$^{\text{r}}$. $2\cdot4\cdot6$

	I	II	III
1	x^9+x^{11}	$x^9+x^{11}+x^{13}$	$-x^{13}$
2	$x^{12}+x^{14}+x^{16}$	$x^{12}+x^{14}+x^{16}$	0
3	$x^{15}+x^{17}+x^{19}+x^{21}$	$x^{15}+x^{17}+x^{19}$	x^{21}
4	$x^{18}+x^{20}+x^{22}+x^{24}+x^{26}$	$x^{18}+x^{20}+x^{22}$	$x^{24}+x^{26}$
5	$x^{21}+x^{23}+x^{25}+x^{27}+x^{29}+x^{31}$	$x^{21}+x^{23}+x^{25}$	$x^{27}+x^{29}+x^{31}$
etc.	etc.	etc.	etc.

The products (4, 2) yield the second column, in which the successive rows are the terms of the expansion

$$\frac{x^9}{2\cdot2\cdot3\cdot4}=\frac{x^9}{2\cdot2\cdot4}+\frac{x^{12}}{2\cdot2\cdot4}+\frac{x^{15}}{2\cdot2\cdot4}+\text{etc.}$$

after reduction to the common denominator $2\cdot4\cdot6$. The reasoning of the preceding article applies here, *mutatis mutandis*, and need not be repeated. The third column is found, as before, by subtraction and gives the following reduced formula for Sextic Perpetuants.

$$(6)+(3,3)=\frac{x^{21}}{2\cdot3\cdot4\cdot5\cdot6}+\left(S_6-\frac{x^{13}}{2\cdot4\cdot6}\right). \tag{A}$$

The syzygies denoted by the last term are easily shown to be identical with the special class of sextic syzygies before mentioned, given by the general formula

$$Q_{2\alpha}\Phi_{\beta,\gamma}-Q_{2\beta}\Phi_{\alpha,\gamma}+Q_{2\lambda}\Phi_{\alpha,\beta}=0,$$

and having the same generating function (see art. 4).

For, from the way in which the reduction has been effected, it is evident that the syzygies in question contain no products of either of the types (2, 2, 2) or (3, 3), but simply connect those of the (4, 2) products in which only one odd part occurs with one another and with forms of lower degree. In the general formula these forms of lower degree are contained in the forms Φ; thus $\Phi_{2,1} = 43 + 322$, and so for others.

Moreover, if we call a (4, 2) syzygy one which does not contain any (3, 3) product, all the (4, 2) syzygies are contained in the above general formula. For it is readily seen that the capitation of an identity, or of a syzygy which contains a (3, 3) product, can never give rise to a (4, 2) syzygy; so that the complete set of these is obtained by first capitating all the Quintic Syzygies with 4.2 throughout, and then capitating the syzygies thus formed with 4.2 throughout, and repeating the process indefinitely. We shall thus find, for the generating function of (4, 2) syzygies, the generating function for Quintic Syzygies multiplied by $\frac{x^6}{6}$; which is precisely the result obtained previously, proving that all the (4, 2) syzygies have been found.

7. The removal of the (3, 3) products from the reduced formula alone remains to be performed, and when this is done we shall obtain at the same time the generating function for Sextic Perpetuants and that for Sextic Syzygies.

The result of removing a single (3, 3) product, symbolized by x^n, is either

$$\left(\frac{x^{21}}{2\cdot3\cdot4\cdot5\cdot6} - x^n\right) + \left(S_6 - \frac{x^{13}}{2\cdot4\cdot6}\right),$$

or

$$\frac{x^{21}}{2\cdot3\cdot4\cdot5\cdot6} + \left(S_6 - x^n - \frac{x^{13}}{2\cdot4\cdot6}\right),$$

where in the former $-x^n$ denotes a reduction of the partitions, and in the latter a syzygy. Hence we see that any (3, 3) product must either give a reduction of the partitions contained in $\frac{x^{21}}{2\cdot3\cdot4\cdot5\cdot6}$ or else a syzygy.

Now the generating function of (3, 3) is $\frac{x^6 + x^{11}}{2\cdot3\cdot4\cdot6}$, and the effect of the reduction of Article 6 on the partitions denoted by

$$\frac{x^6}{2\cdot3\cdot4\cdot5\cdot6} = \frac{x^6 + x^{11} + x^{16} + x^{21} + x^{26} + x^{31} + x^{36} + x^{41} + \text{etc.}}{2\cdot3\cdot4\cdot6}$$

has been to remove the terms $x^6 + x^{11} + x^{16}$ from the numerator of this last fraction; so that it would appear at first sight that no fresh reduction is given by any of the (3, 3) products, in which case all of them would correspond to syzygies, this however is known to be untrue. But if we write the generating function of (3, 3) in the equivalent form

$$\frac{x^6 + x^9 + x^{11} + x^{12} + x^{14} + x^{15} + x^{17} + x^{18} + x^{20} + x^{23}}{2\cdot4\cdot6} + \frac{x^{21} + x^{26}}{2\cdot3\cdot4\cdot6}$$

we see that the next two terms $x^{21}+x^{26}$ may also be removed from the numerator, leaving $x^{31}=x^{36}+x^{41}+$ etc. The unreduced partitions are now

$$(6)=\frac{x^{31}}{2\cdot3\cdot4\cdot5\cdot6}.$$

That portion of the generating function for (3, 3) which does not give reductions will correspond to syzygies; and hence we have, introducing the term x^{13} in the numerator to represent the (4, 2) syzygies,

$$S_6=\frac{x^6+x^9+x^{11}+x^{12}+x^{13}+x^{14}+x^{15}+x^{17}+x^{18}+x^{20}+x^{23}}{2\cdot4\cdot6}.$$

To complete the proof it would be necessary to identify this with the complete set of Sextic Syzygies, which would involve the calculation of such of them as have not already been found by Capt. MacMahon. The above value of S_6 may, by introducing common factors in its numerator and denominator, be written in either of the equivalent, but more concise, forms

$$S_6=\frac{x^6+x^{11}+x^{13}-x^{16}-x^{21}-x^{26}}{2\cdot3\cdot4\cdot6}=\frac{x^6+x^{13}-2x^{16}-x^{18}+x^{31}}{2\cdot3\cdot4\cdot5\cdot6},$$

of which the second has been given by Prof. Cayley.

8. The formulæ for the seventh degree are

$$(5,2)=\frac{x^{17}}{2\cdot2\cdot3\cdot4\cdot5},\ (4,3)=\frac{x^{10}}{2\cdot2\cdot3\cdot3\cdot4},\ (3,2,2)=\frac{x^{7}}{2\cdot2\cdot3\cdot4},$$

$$(7)+(5,2)+(4,3)+(3,2,2)=\frac{x^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7}+S_7,$$

by means of which we may verify the relation

$$(7)+(4,3)=\frac{x^{28}}{2\cdot3\cdot4\cdot5\cdot6\cdot7}+\left(S_7-\frac{x^9+x^{11}}{2\cdot4\cdot6}\right),$$

but remembering that $S_7=S_7^{(1)}+(2)\,S_5^{(1)}-S_7^{(2)}$ (see Article 2), and substituting for $S_5^{(1)}$ and $S_7^{(2)}$ their values given in Article 4; this reduces to

$$\text{(B)}\qquad (7)+(4,3)=\frac{x^{28}}{2\cdot3\cdot4\cdot5\cdot6\cdot7}+S_7^{(1)},$$

indicating that every ground-syzygy of the seventh degree contains at least one term of the (4, 3) type. This theorem was originally obtained by the method employed in Articles 5 and 6, and there is every reason to believe that a tolerably simple independent proof of it is possible; but my present object is briefly to indicate the formulæ for Perpetuants of the seventh degree and to consider in the remaining articles the case of Finite Quantics. We have

$$(4,3)=\frac{x^{10}+x^{12}+x^{13}+x^{14}+x^{15}+x^{16}+x^{17}+\ldots}{2\cdot3\cdot4\cdot6}$$

$$=\frac{N}{2\cdot3\cdot4\cdot6}+\frac{x^{28}+x^{35}+x^{42}+x^{49}+x^{56}}{2\cdot3\cdot4\cdot5\cdot6}$$

where $N=x^{10}+x^{12}+x^{13}+x^{14}+x^{15}+x^{16}+x^{17}+x^{18}+x^{19}+x^{20}+x^{21}+x^{22}+x^{23}$
$+x^{24}+x^{25}+x^{26}+x^{27}+x^{29}+x^{30}+x^{31}+x^{32}+x^{34}+x^{36}+x^{37}+x^{39}$
$+x^{41}+x^{44}+x^{46}+x^{51}$.

And if we write

$$\frac{x^{28}}{2\cdot3\cdot4\cdot5\cdot6\cdot7}=\frac{x^{28}+x^{35}+x^{42}+x^{49}+x^{56}+x^{63}+x^{70}+x^{77}+\text{etc.}}{2\cdot3\cdot4\cdot5\cdot6}$$

the method of Article 7 will give

$$(7)=\frac{x^{63}}{2\cdot3\cdot4\cdot5\cdot6\cdot7},$$

and $$S_7^{(1)}=\frac{N}{2\cdot3\cdot4\cdot6}=\frac{x^{10}+x^{12}+x^{13}+x^{14}+x^{16}-x^{28}-x^{35}-x^{42}-x^{49}+x^{56}}{2\cdot3\cdot4\cdot5\cdot6};$$

where the second value of $S_7^{(1)}$ is obtained from the first by introducing the common factor $1-x^5$ in both numerator and denominator. The first value would be more convenient than the second if we had in our possession a complete list of the ground-syzygies and wished to compare them with the indicated value of $S_7^{(1)}$. Such a comparison would be in itself a most complete and satisfactory proof of the results given in this article, but would require some very laborious calculations. The terms $x^{10}+x^{12}+x^{14}$ of N are easily seen to correspond to a set of syzygies of the form $4^{\alpha+1}3^{\beta+1}2^{\gamma}.\,32^{\delta}-54^{\alpha}3^{\beta+1}2^{\gamma}.\,2^{\delta+1}\equiv0$; but I have not attempted to compare any of the remaining twenty-six terms with syzygies.

Two different reductions have been made use of in these articles; the first of these corresponds to the reduction of the generating function of a Finite Quantic to its Representative form, and consists in removing all terms which contain Quadric Factors; the second reduction removes all terms with Cubic Factors. For the eighth degree the formula after the first reduction is

$$\text{(C)}\quad (8)+(5,3)+(4,4)=\frac{x^{36}}{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}+S_8^{(1)}+(3)S_5^{(1)}+\left(\frac{x^{21}}{2\cdot4\cdot6\cdot8}-S_8^{(2)}\right),$$

the terms (6, 2), (4, 2, 2), (3, 3, 2), (2, 2, 2, 2), and $(2)S_6^{(1)}$, which contain Quadric Factors, having been removed. The term whose numerator is x^{21} corresponds to the "special class of $S_8^{(2)}$ syzygies" mentioned in Article 3. This reduction does not assume the knowledge of either (6) or $S_6^{(1)}$, but only of their difference; the result is therefore unaffected by any error in the determination of (6), should any such error exist.

9. To the formula (5) $= \frac{x^{15}}{2\cdot3\cdot4\cdot5}$ and to (A), (B), and (C) of the preceding articles may be added one for the ninth degree, viz.

$$\text{(D)}\quad (9)+(6,3)+(5,4)+(3,3,3)=\frac{x^{45}}{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9}+S_9^{(1)}+(3)\,S_6^{(1)}+(4)\,S_5^{(1)}-S_9^{(2)}$$

but the proof of this formula, as well as that of (C), has been omitted for the sake of brevity.

Observe (1) that in each of them all terms containing a Quadric Factor have been removed from both sides of the equation.

(2) That the expressions found in Article 4 for $S_5^{(1)}$, $S_7^{(2)}$, and $S_9^{(3)}$ have been substituted for them in the formulæ for degrees 5, 7, and 9 respectively.

(3) That each of them contains a term of the form

$$\frac{x^{\frac{n(n+1)}{2}}}{2\cdot3\cdots n},\ \text{where } n=5,\ 6,\ 7,\ 8,\ 9.$$

This term alone expresses the unreduced partitions, viz. if the partition expressing any Quintic Perpetuant be $5^{\delta+1}4^{\gamma+1}3^{\beta+2}2^{\alpha}$, the unreduced partitions in (A), (B), (C) and (D) may be found by the successive superaddition of $\varepsilon+1$ parts each $=6$, $\zeta+1$ parts each $=7$, $\eta+1$ parts each $=8$, and $\theta+1$ parts each $=9$; so that the unreduced partitions in (D) will be of the form $9^{\theta+1}8^{\eta+1}7^{\zeta+1}6^{\epsilon+1}5^{\delta+1}4^{\gamma+1}3^{\beta+2}2^{\alpha}$.

Applications to the Theory of Finite Quantics.

10. In a recent number of the *American Journal of Mathematics* (Vol. VII, p. 337) I found for the Generating Function of any Binary Quantic the expression

$$\frac{1-\Sigma_1+\Sigma_2-\Sigma_3+\ldots.}{\Pi(1-a^r x^s)}$$

where $(r.s)$ is the deg. order of any groundform, and the product Π contains a factor corresponding to each groundform. If there be two groundforms of the same deg. order Π will of course contain a corresponding square factor, if three a cubic factor, and so on. In the numerator Σ_n is the aggregate of terms corresponding to the irreducible syzygies of the n^{th} grade, or say the irreducible n^{th} syzygies; and the portion of the generating function corresponding to the portion Σ_n of the numerator will, when expanded, give a series of terms corresponding to all the n^{th} syzygies reducible or irreducible. The generating function when written in this form serves to express the fact that, for any given deg. order, the number of asyzygetic covariants is equal to the total number of covariants minus the total number of first syzygies, of that deg. order, plus the total number of second syzygies minus the total number of third syzygies, and so on.

It may now be easily shown that the numerator of any Representative form of Generating Function for a Binary Quantic will consist of blocks of alternately positive and negative terms; the terms in the first block representing covariants, which may be either irreducible or compound, but none of the compounds are divisible by any covariant represented in the denominator; the terms in the second block, in like manner, representing first syzygies not divisible by any of the denominator covariants; those in the third block second syzygies, and so on. To see this it is only necessary to replace all the factors $(1-a^r x^s,\ 1-a^{r'} x^{s'} \dots)$ of Π which do not occur in the denominator of the representative form under consideration by geometrical series multiplying its numerator, which will therefore be $(1-\Sigma_1+\Sigma_2-\Sigma_3+\dots)\times(1+a^r x^s+x^{2r}x^{2s}+\dots)(1+a^{r'}x^{s'}+a^{2r'}x^{2s'}+\dots)\dots$; viz. the compound covariants of the first block will simply be powers and products of $(r.s)$, $(r'.s')$, ...; and the compound syzygies of the subsequent blocks will be simply divisible by these powers and products. If after multiplying by these geometrical series, we allow terms such as $+a^m x^n$ and $-a^m x^n$ to cancel each other,* there will result a Representative Generating Function of the form

$$\frac{C-S_1+S_2-S_3+\dots}{D};$$

in which D is a representative divisor of Π, *i. e.* it is the product of factors $(1-a^p x^q)(1-a^{p'} x^{q'})$ . . . representing groundforms whose deg. orders are $(p.q)$, $(p'.q')$, ...; and of the covariants in C, or the n^{th} syzygies in S_n, those which are reducible will be divisible by powers and products of the groundforms $(r.s)$, $(r's')$ . . . , but not divisible by any of the groundforms $(p.q)$, $(p'q')$, . . . represented by the factors of D.

11. Dr. F. Franklin, in a paper "On the Calculation of the Generating Functions and Tables of Groundforms for Binary Quantics" (*American Journal*

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*Positive and negative terms may in most cases be freely allowed to destroy one another, for a $(n+1)^{\text{th}}$ syzygy may be considered as a relation between the *compound* n^{th} syzygies diminishing their number. But subsequent corrections will have to be made, not only for cases of the coexistence of groundform and syzygy, but also wherever $(n+1)^{\text{th}}$ syzygies exist simultaneously with *irreducible* n^{th} syzygies of the same deg. order. Examples of this do in fact occur even for the Binary Quintic, which has irreducible first and second syzygies of the same deg. order.

of Mathematics, Vol. III, p. 137), has given the numerator of the representative generating function whose denominator is

$$D = (1 - a^p x^q)(1 - a^{p'} x^{q'})(1 - a^{p''} x^{q''}) \ldots$$

in the form

$$L\,(1 - \Sigma a^p x^q + \Sigma a^{p+p'} x^{q+q'} - \Sigma a^{p+p'+p''} x^{q+q'+q''} + \ldots),$$

where L is the expansion of the generating function in an infinite series, and is multiplied by the finite expanded value of D which forms the series in brackets. He also proves that the covariants represented by terms of the numerator are not divisible by any of the denominator forms $(p.q)$, $(p'.q')$, $(p''.q'')$, ... This form of numerator will of course reduce to that previously given, viz. $C - S_1 + S_2 - S_3 + \ldots$, if we freely allow positive and negative terms to destroy one another; but among the terms of $L\Sigma a^p x^q$ some will correspond to compound covariants of the type $(p.q)\Theta$, where Θ is one of the asyzygetic forms represented by the terms of L, and will properly be subtracted from L in order to reduce it to C; others, *which cannot thus be subtracted*, will correspond to syzygies, each of which will give an expression for a compound of the type $(p.q)\Theta$ in terms of the asyzygetic compounds. So also, some of the terms of $L\Sigma a^{p+p'} x^{q+q'}$ will correspond to compound syzygies, reducing the syzygies of the preceding block to S_1, and some of them to "second syzygies"; and reductions will take place, in like manner, in each of the alternately positive and negative blocks of the numerator.

But since positive and negative terms have been freely allowed to cancel each other,* no term in S_1 is of the same deg. order as a term in C; thus the syzygy giving $(p.q)\Theta$ is not of the same deg. order as any term of C, and since the asyzygetic compounds not contained in C are all of them divisible by one or more of the denominator forms $(p.q)$, $(p'.q')$, $(p''.q'')$, ..., each of the syzygies S_1 will connect only compound covariants which are divisible thus.

This reasoning may, without difficulty, be extended so as to give the meaning of terms in the remaining blocks S_2, S_3, ..., when it will be seen that each term of the block S_n denotes a n^{th} syzygy not itself divisible by any denominator form but connecting only such compound $(n - 1)^{\text{th}}$ syzygies as are so divisible. If we call a, $Q_{2\alpha}$, $Q_{2\beta}$, $Q_{2\gamma}$, ... denominator forms, the first, second, and third syzygies whose expressions have been given in Article 3 fulfil these conditions.

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* Subsequent corrections will of course have to be made for such cases of coexistence as are mentioned in the preceding note.

12. Referring to p. 135 of Dr. Franklin's paper, just quoted, it will be seen that the representative function will be expressed in a finite form whenever its denominator D is divisible by the denominator of the reduced form, but in no other case. For a given quantic there is sometimes a large number of different finite representative generating functions, even when the number of factors in the denominator is restricted. As an example, for the Quintic there is no finite representative generating function with less than six factors in its denominator, but as many as 18 can be found with six factors. Here the denominator of the reduced form is

$$(1-a^4)(1-a^6)(1-a^8)(1-ax)(1-ax^3)(1-ax^5),$$

the denominator of the simplest representative form being

$$(1-a^4)(1-a^8)(1-a^{12})(1-a^2x^2)(1-a^2x^6)(1-ax^5);$$

but the three invariant factors of the denominator may be chosen in 3 different ways, viz.

$$(1-a^4)(1-a^8)(1-a^{12})$$
$$(1-a^4)(1-a^8)(1-a^{18})$$
$$(1-a^8)(1-a^{12})(1-a^{18}),$$

and similarly the three covariant factors may be chosen in 6 different ways; or there are in all 18 different ways. For one of the covariant factors must be $(1-ax^5)$, and denoting the other two by $(1-a^px^q)(1-a^rx^s)$, $(p.q)$ may be either (2.2), (3.3), or (4.4), and $(r.s)$ may be either (2.6) or (3.9).

In Prof. Sylvester's very extensive and extremely valuable Tables of Generating Functions and Groundforms, published in various numbers of the *American Journal of Mathematics*, the representative form given is in each case the simplest that can be found by multiplying both numerator and denominator of the reduced form, given in the same table, by such a factor as will make the denominator, and therefore the numerator, representative. But since no two representative forms give us exactly the same information, it may in some cases be convenient, instead of those given in the tables, to use other forms; and any one of these may be obtained from the reduced form by the same process.

Everything in this and in the two preceding articles may be either directly applied or easily extended to Systems of Binary Quantics, but in what follows the quantic $(a, b, c, \ldots \between x, y)^n$, and its simplest representative function, will alone be considered when $n = 5, 6, 7, 8, 9, 10, 12$.

13. The *source* of any covariant is what remains of it after putting $x = 1$ and $y = 0$, and is a *perpetuant* when not expressible as a rational integral function of the sources of covariants of lower deg. order, however great the order of the parent quantic may be.

A syzygy containing non-perpetuant sources may be a non-perpetuant syzygy in two distinct cases: (1) after substituting for each non-perpetuant source its value in terms of perpetuants, the syzygy may be expressible as a linear function of the compound syzygies to the quantic of infinite order; (2) this substitution may cause the syzygy to vanish. Irreducible syzygies to a quantic of finite order, when not included in either of these classes, are also *perpetuant syzygies.*

Since all groundforms of degrees 1, 2, and 3, have perpetuant sources, and since there are no perpetuant syzygies of lower degree than 5; no single binary quantic has a syzygy of lower degree than 5, and all its syzygies of degree 5 are perpetuant syzygies. For, all compound sources of degree 4 are made up of perpetuant sources of degrees 1, 2, and 3; so that no syzygy of degree 4 can exist which contains a non-perpetuant source, *i. e.* there is no non-perpetuant syzygy.

Syzygies of degree 5 to any binary quantic, besides terms divisible by the source of the quantic itself, or a, contain terms formed by multiplying a source of degree 2 by one of degree 3; or denoting these by $(2)(3)$, $(2)'(3)'$, ... we may write any syzygy of degree 5 in the form of a congruence, thus

$$\lambda\,(2)(3) + \mu\,(2)'(3)' + \ldots \equiv 0 \pmod{a},$$

where λ, μ, ... are numerical multipliers to be determined.

But this is a perpetuant congruence, and corresponds to a perpetuant syzygy. Again, there are no second syzygies of lower degree than 7; for there are no compound syzygies of degree 5, and therefore no second syzygies of that degree, and denoting the syzygies of degree 5 by $(\bar{5})$, $(\bar{5})'$, ..., the only compound syzygies of degree 6 are $a(\bar{5})$, $a(\bar{5})'$, ... which are clearly not connected by a second syzygy. For degree 7 the compound syzygies not divisible by a are $(2)(\bar{5})$, $(2)'(\bar{5})'$, ..., so that any second syzygy of degree 7 may be written as a congruence, thus $\lambda\,(2)(\bar{5}) + \mu\,(2)'(\bar{5})' + \ldots \equiv 0 \pmod{a}$, corresponding to a perpetuant second syzygy, of which the general value has been given in Article 3.

By extending this reasoning to the 3d, 4th, 5th, ... syzygies in succession, we may prove that there are no n^{th} syzygies of lower degree than $2n + 3$, and that the n^{th} syzygies of this degree are perpetuant n^{th} syzygies, so that their number is given by the partition theorem of Article 3. And it should be remembered that every n^{th} syzygy thus found contains a $(n - 1)^{\text{th}}$ syzygy whose weight is the same, its degree being less by unity than that of the n^{th} syzygy.

14. The number of partitions of n into $\lambda + 1$ unequal parts, zeros excluded and no part greater than μ, is the coefficient of $c^{\lambda+1}z^n$ in

$$(1 + cz)(1 + cz^2)(1 + cz^3)\ldots(1 + cz^\mu) - (1 + cz + cz^2 + cz^3 + \ldots + cz^\mu)$$

when λ has any positive integral value from 1 to μ inclusive.

But by the theorem of Article 3 this is the number of λ^{th} syzygies of degree $2\lambda + 3$ and weight $2n + 1$, to the quantic of order i; where $i - 1 = 2\mu$ or $i - 2 = 2\mu$, according as i is odd or even.

Hence we may, if $w = 2n + 1$, instead of the above expression, use

$$\frac{ax}{g}(1 + a^2gx^2)(1 + a^2gx^4)\ldots(1 + a^2gx^{2\mu}) - \frac{ax}{g}(1 + a^2gx^2 + a^2gx^4 + \ldots + a^2gx^{2\mu}),$$

in which the coefficient of $a^{2\lambda+3}g^\lambda x^w$ is the number of λ^{th} syzygies of degree $2\lambda + 3$ and weight w, to either the quantic of order $2\mu + 1$ or that of order $2\mu + 2$. Thus, for the Quintic or Sextic,

$$\frac{ax}{g}(1 + a^2gx^2)(1 + a^2gx^4) - \frac{ax}{g}(1 + a^2gx^2 + a^2gx^4) = a^5gx^7$$

represents the well-known syzygy of degree 5 and weight 7, which is the same for both quantics.

For the 7^{ic} or 8^{ic} we have

$$\begin{aligned}\frac{ax}{g}(1 + a^2gx^2)(1 + a^2gx^4)(1 + a^2gx^6) &- \frac{ax}{g}(1 + a^2gx^2 + a^2gx^4 + a^2gx^6)\\ &= a^5g(x^7 + x^9 + x^{11}) + a^7g^2x^{13},\end{aligned}$$

so that these two quantics have one syzygy of degree 5 for each of the weights 7, 9 and 11, and a second syzygy of degree 7 and weight 13. It is to be remembered that this second syzygy contains a syzygy of deg. weight 6.12 the only representative of the special class mentioned in Article 3.

For the 9^{ic} or 10^{ic} we have in like manner

$$\begin{aligned}&\frac{ax}{g}(1+a^2gx^2)(1+a^2gx^4)(1+a^2gx^6)(1+a^2gx^8) - \frac{ax}{g}(1+a^2gx^2+a^2gx^4+a^2gx^6+a^2gx^8)\\ &\quad= a^5g(x^7 + x^9 + 2x^{11} + x^{13} + x^{15}) + a^7g^2(x^{13} + x^{15} + x^{17} + x^{19}) + a^9g^3x^{21},\end{aligned}$$

which must be interpreted as before.

And for the 11^{ic} or 12^{ic},

$$\begin{aligned}&\frac{ax}{g}(1 + a^2gx^2)(1 + a^2gx^4)(1 + a^2gx^6)(1 + a^2gx^8)(1 + a^2gx^{10})\\ &\qquad\qquad - \frac{ax}{g}(1 + a^2gx^2 + a^2gx^4 + a^2gx^6 + a^2gx^8 + a^2gx^{10})\\ &= a^5g(x^7 + x^9 + 2x^{11} + 2x^{13} + 2x^{15} + x^{17} + x^{19})\\ &+ a^7g^2(x^{13} + x^{15} + 2x^{17} + 2x^{19} + 2x^{21} + x^{23} + x^{25})\\ &+ a^9g^3(x^{21} + x^{23} + x^{25} + x^{27} + x^{29})\\ &+ a^{11}g^4x^{31}.\end{aligned}$$

The above is a complete determination of the syzygies of degree 5, second syzygies of degree 7, third syzygies of degree 9, and fourth syzygies of degree 11, for the quantics considered. A partial determination of the syzygies of degree 6, second syzygies of degree 8, and third syzygies of degree 10, for these quantics, will be obtained by simply dividing those terms that contain any power of g higher than the first by ag, thus for the 9^{ic} or 10^{ic} we have

$$a^6 g\,(x^{13} + x^{15} + x^{17} + x^{19}) + a^8 g^2 x^{21}.$$

As was shown in Article 11, the successive positive and negative blocks in the numerator of any representative generating function, for which the denominator forms are a, Q_2, Q_4, ... ought to contain terms corresponding to all of these syzygies, and when this is not so, the corrections spoken of in the foot-notes of Articles 10 and 11 will have to be made.

15. To illustrate the nature of these corrections, take the case of the 12^{ic}. Here writing $g = -1$ and replacing deg. weight by deg. order, the odd-degree formula becomes

$$\begin{aligned}
&- a^5\,(x^{22} + x^{26} + 2x^{30} + 2x^{34} + 2x^{38} + x^{42} + x^{46})\\
&+ a^7\,(x^{34} + x^{38} + 2x^{42} + 2x^{46} + 2x^{50} + x^{54} + x^{58})\\
&- a^9\,(x^{50} + x^{54} + x^{58} + x^{62} + x^{66})\\
&+ a^{11} x^{70}\,;
\end{aligned}$$

while the even-degree formula will be found to be

$$\begin{aligned}
&- a^6\,(x^{22} + x^{26} + 2x^{30} + 2x^{34} + 2x^{38} + x^{42} + x^{46})\\
&+ a^8\,(x^{38} + x^{42} + x^{46} + x^{50} + x^{54})\\
&- a^{10} x^{58}\,;
\end{aligned}$$

viz. the latter is in this case obtained from the former by dividing the terms of degrees 7, 9, and 11 by $-ax^{12}$ instead of by ag.

These are to be compared with the following terms selected from the numerator of Prof. Sylvester's representative generating function for the 12^{ic} (*American Journal of Mathematics*, Vol. IV, pp. 44, 45).

$$\begin{aligned}
&a^5\,(4x^{22} + x^{24} + x^{26} - x^{30} - 2x^{34} - 2x^{38} - x^{42} - x^{46})\\
&a^6\,(6x^{22} + 2x^{24} + x^{26} - x^{30} - 3x^{34} - 2x^{38} - x^{42} - x^{46})\\
&a^7\,(- 4x^{34} - x^{38} + x^{42} + 2x^{46} + 2x^{50} + x^{54} + x^{58})\\
&a^8\,(- 2x^{38} - x^{40} + 2x^{46} + x^{50} + x^{54})\\
&a^9\,(x^{50} - x^{58} - x^{62} - x^{66})\\
&- a^{10} x^{58}\\
&+ a^{11} x^{70}
\end{aligned}$$

where the terms of the 5th and 6th degrees in a have been taken partly from the first positive, and partly from the first negative block; those of the 7th and 8th degrees partly from the first negative, and partly from the second positive block; those of the 9th degree partly from the second positive, and partly from the second negative, $-a^{10}x^{58}$ from the second negative, and $a^{11}x^{70}$ from the third positive block.

Now writing, as in Articles 10 and 11, the numerator in the form

$$C - S_1 + S_2 - S_3 + S_4,$$

we have

$$\begin{aligned}
C &= \ldots . + a^5(\ldots + 5x^{22} + x^{24} + 2x^{26} + x^{30}) \\
&\quad + a^6(\ldots + 7x^{22} + 2x^{24} + 2x^{26} + x^{30}) + \ldots \\
-S_1 &= -a^5(x^{22} + x^{26} + 2x^{30} + 2x^{34} + 2x^{38} + x^{42} + x^{46}) \\
&\quad - a^6(\ldots + x^{22} + x^{26} + 2x^{30} + 3x^{34} + 2x^{38} + x^{42} + x^{46}) \\
&\quad - a^7(\ldots + 5x^{34} + 2x^{38} + x^{42}) \\
&\quad - a^8(\ldots + 3x^{38} + x^{40} + x^{42}) - \ldots \\
S_2 &= a^7(x^{34} + x^{38} + 2x^{42} + 2x^{46} + 2x^{50} + x^{54} + x^{58}) \\
&\quad + a^8(\ldots + x^{38} + x^{42} + 2x^{46} + x^{50} + x^{54}) \\
&\quad + a^9(\ldots + 2x^{50} + x^{54}) + \ldots \\
-S_3 &= -a^9(x^{50} + x^{54} + x^{58} + x^{62} + x^{66}) \\
&\quad - a^{10}(\ldots + x^{58}) - \ldots . \\
S_4 &= a^{11}x^{70} + \ldots .
\end{aligned}$$

which are expressions for the separated blocks. It will be seen, if we recombine these, and thus return to the original form of the numerator, that contiguous blocks have a tendency to overlap one another; what I have called, for want of a better word, *corrections* separate them again and restore each term to its proper block. The restorations depending on the odd-degree formulæ of the last article may be relied on as exact, those depending on the even-degree formulæ can only be considered as approximations to an exact restoration.

16. Passing on to the consideration of the first block C in the numerator of any representative function which contains, among its denominator forms, the quantic itself and all its quadricovariants, including invariants; it will be seen at once that since none of the covariants represented by terms of C are divisible by any of the denominator forms, C cannot contain any terms of the first or second degree, and all its terms of the third, fourth, and fifth degrees will represent groundforms. It may contain terms of the sixth degree representing compounds obtained by multiplying together a pair of cubic groundforms; and the terms of higher degree that it may contain may be found from a table of the partitions of

any number n from which the parts 1 and 2 have been removed;* thus

n	Partitions
1	None
2	None
3	3
4	4
5	5
6	6, 3.3
7	7, 4.3
8	8, 5.3, 4.4
9	9, 6.3, 5.4, 3.3.3
etc.	. . . .

Since there are no syzygies of any kind of lower degree than the fifth, the groundforms of the first, second, third, and fourth degrees may be found by simply adding the denominator forms of these degrees to those represented by terms in C, and will always be identical with those obtained by Tamisage.†

Since no terms of the 5th degree occur in S_2, S_3, . . . , the numerator N may be written $N = C - S_1$ if we reject all terms of higher and all of lower degree than the fifth; and then the theory is that all the groundforms, except those in the denominator, and no compounds are represented by the terms of C, and that all the syzygies are perpetuant syzygies, which are all represented by terms of S_1. To obtain the complete list of groundforms of the fifth degree to the quantic of order i, we take the value of S_1 from the following table:

i	S_1
5	a^5x^{11}
6	a^5x^{16}
7	$a^5(x^{13} + x^{17} + x^{21})$
8	$a^5(x^{18} + x^{22} + x^{26})$
9	$a^5(x^{15} + x^{19} + 2x^{23} + x^{27} + x^{31})$
10	$a^5(x^{20} + x^{24} + 2x^{28} + x^{32} + x^{36})$
11	$a^5(x^{17} + x^{21} + 2x^{25} + 2x^{29} + 2x^{33} + x^{37} + x^{41})$
12	$a^5(x^{22} + x^{26} + 2x^{30} + 2x^{34} + 2x^{38} + x^{42} + x^{46})$
etc.	. . . .

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* It may possibly be useful at this point to recall the corresponding formulæ in the theory of Perpetuants, viz. (A), (B), (C), and (D) of Articles 6, 8, and 9.

† In the denominator for the Nonic (*American Journal of Mathematics*, Vol. II, p. 236) the printer has omitted the factor $(1 - a^2x^2)$.

the value of C is then found by adding S_1 to the value of N taken from Prof. Sylvester's Tables of Generating Functions and Groundforms, in the pages of this Journal, to which reference is made; and finally, the denominator forms of degree 5, if there are any, are added.

Thus when $i = 7$ (Vol. II, p. 228)

$$N = a^5(x + 2x^3 + 2x^5 + 2x^7 + 2x^9 - x^{17} - x^{21})$$

$$C = a^5(x + 2x^3 + 2x^5 + 2x^7 + 2x^9 + x^{13})$$

there is no denominator form of degree 5, and all the groundforms are represented by terms of C; these are the same as given in the table p. 230, with the additional groundform of deg. order (5.13).

When $i = 8$ (Vol. II, p. 232)

$$N = a^5(x^2 + 2x^4 + 2x^6 + x^8 + 3x^{10} + x^{14} - x^{18} - x^{22} - x^{26})$$

the addition of S_1 to this simply destroys the negative terms and there is no additional groundform; but there is a denominator form (the Quintinvariant) a^5, and this added to C gives the expression

$$a^5(1 + x^2 + 2x^4 + 2x^6 + x^8 + 3x^{10} + x^{14})$$

corresponding to the complete set of quintic groundforms given in the table (p. 233). These two examples will sufficiently explain the way in which the following table is calculated

	i	Additional Groundforms
Vol. II, p. 224	5	None
" 225	6	None
" 230	7	a^5x^{13}
" 233	8	None
" 242	9	$a^5(x^{15} + x^{19})$
" 247	10	a^5x^{20}
Not calculated	11	—
Vol. IV, p. 48	12	$a^5(x^{22} + x^{26} + x^{30})$

Some of these were found by a different method as long ago as the beginning of 1883; there are also additional Groundforms for the 11ic which are not given here, but may be found without much difficulty (See the J. H. U. Circulars, No. 22, April, 1883).

17. The success of the method of the last article was dependent on the combination of three accidents, viz.

(1) All the Numerator Covariants were Groundforms.
(2) All the Numerator Syzygies were Perpetuant Syzygies.
(3) All the Perpetuant Syzygies were Numerator Syzygies.

These will not happen for the sixth or higher degrees.

Example 1. The Quintic has six perpetuant syzygies of degree 6, but not one of them is represented by a term in the Numerator of the Representative Function, which (see Vol. II, p. 224) has no negative term of the sixth degree.

Example 2. In the Numerator of the Representative Function for the Sextic (Vol. II, p. 225) the term $+a^6x^4$ represents the square of the covariant of deg. order (3.2). The products $(2.0)^2(2.4)$, $(2.0)(4.4)$, $(2.4)(4.0)$ are not Numerator Covariants because each of them is divisible by a denominator form.

Example 3. The syzygy $an - b^2e + 6bl + 2ck + fg = 0$ of deg. order (7.9) for the Quintic (see Prof. Cayley's 10th Memoir on Quantics), is a Perpetuant Syzygy, and a Numerator Syzygy for the Quintic because, though the syzygy itself is not, each of its terms is divisible by one of the denominator forms (a, b, c, g, q, u); but it is not a Numerator Syzygy for the Sextic, because, though a, b, c remain denominator forms for the Sextic, neither f nor g is a denominator form. In fact f is the source of the covariant (3, 12) and g of the covariant (4.4), both of which are numerator forms for the Sextic.

Example 4. Of the two negative terms $-a^6(x^{14}+x^{16})$ in the Numerator for the Septimic (Vol. II, p. 228), one represents a Perpetuant Syzygy, the other a non-perpetuant Syzygy.

For the syzygies of the fifth degree we may write

Deg. Order.
(5.13) $(3.7)(2.6) - (3.11)(2.2) = (1.7)(4.6)$
(5.17) $(3.7)(2.10) - (3.15)(2.2) = (1.7)(4.10)$
(5.21) $(3.11)(2.10) - (3.15)(2.6) = (1.7)(4.14)$.

Multiplying the first of these by (2.10), the second by $-(2.6)$, the third by (2.2), and adding, we deduce

(6.16) $(2.10)(4.6) - (2.6)(4.10) + (2.2)(4.14 = 0$,

which is one of the perpetuant syzygies in question.

It should be noticed that the four are so connected that when any three of them are known the fourth can be found by means of the second syzygy.

(7.23)
$$\begin{aligned}&(2.10)\{(3.7)(2.6) - (3.11)(2.2) - (1.7)(4.6)\}\\ -&(2.6)\ \{(3.7)(2.10) - (3.15)(2.2) - (1.7)(4.10)\}\\ +&(2.2)\ \{(3.11)(2.10) - (3.15)(2.6) - (1.7)(4.14)\}\\ +&(1.7)\ \{(2.10)(4.6) - (2.6)(4.10) + (2.2)(4.14)\} = 0,\end{aligned}$$

which could not exist if one of them were missing. Thus the existence of the term $+a^7x^{23}$ in the numerator, besides indicating the second syzygy of deg. order (7.23) implies the existence of the syzygy of deg. order (5.13).

From the following three perpetuants,

$$\Theta = ai - 8bh + \dots$$
$$\Phi = i(ac - b^2) - 4adh + \dots$$
$$\Psi = i(ae - 4bd + 3c^2) - 4afh + \dots,$$

we may obtain, by eliminating i, three sources for the 7^{ic} connected by a syzygy.

Using the letters P, Q, R, to denote these three sources, we have

$$P = (ac - b^2)\Theta - a\Phi$$
$$Q = (ae - 4bd + 3c^2)\Theta - a\Psi$$
$$R = (ae - 4bd + 3c^2)\Phi - (ac - b^2)\Psi$$

connected by the syzygy

$$aR + (ae - 4bd + 3c^2)P - (ac - b^2)Q = 0.$$

It might easily be proved that P, Q, R, when considered as sources for the 7^{ic}, are irreducible; they may therefore be taken for the sources of the three non-perpetuant groundforms of deg. order (4.8), (4.4), and (5.7) respectively, and the above syzygy may be written

$$(1.7)(5.7) + (2.6)(4.8) - (2.10)(4.4) = 0, \tag{6.14}$$

which is the syzygy represented by $-a^6x^{14}$.

Referring to the definitions of Article 13 it is clear that this syzygy belongs to the second kind of non-perpetuant syzygies.

18. I will conclude by stating a theorem which I believe to be true, but have in vain tried to prove.

Every irreducible syzygy must contain among its terms at least one binary combination of the groundforms.

The converse of this, viz.

Every syzygy containing at least one binary combination of the groundforms among its terms is irreducible, is obviously true.